FUNDAMENTALS

INTRODUCTION

Professor Clive Neal-Sturgess

Emeritus Professor of Mechanical Engineering
1985-2005 Jaguar Professor of Automotive Engineering
Director Automotive Safety Centre

What is Mechanics?

Mechanics is the study *motion* and motion needs energy. For the purposes this module it can be taken to mean the interaction between forces and bodies.

.

Note we have not yet defined what a force is! That will come later in the module when we get to Dynamics.

Numerous laws have been proposed over the past few thousand years to allow answers to be given to questions like:

- what set of forces can be applied to a body without causing it to break
- how does an object move when a different set of forces is applied
- How does motion change with time
- what happens when bodies collide, an example of impact

Mechanics is not just for Mechanical Engineers!

All engineers require a basic understanding of mechanics, whatever their precise discipline. That is why it forms part of the core material. A knowledge of mechanics, for example, would be required for the following activities:

- designing a chemical reactor vessel
- designing a bridge
- designing the electrical power unit for a lift
- designing an automobile ABS
- designing a warehouse storage and retrieval system
- designing a particle reinforced metal composite

- **MECHANICS – the study of *motion***
  - **Motion can be either translation or deflection**

- **Motion needs *Energy***

- **∴ Mechanics**
  - **the study of Energy**
  - **the study of how energy is converted from one form to another**

- **POTENTIAL ENERGY**
  - **The ability (potential) to do work**
- **Many forms**
  - **Mechanical**
    - **Gravitational Potential Energy =PE**
    - **Motion (kinetic energy) = KE**
  - **Electrical**
  - **Chemical**

- **WHERE DOES IT COME FROM?**
  - **The sun**

THE UNIVERSAL LAW OF MOTION & POSITION

"PRINCIPLE OF LEAST ACTION"

“When a change occurs in nature, the quantity of action necessary for the change is the least possible”

(M. de Maupertuis Lyon 1756)

“People were using the Principle of Least Action before they knew what it was”

(Wittgenstein: Tractus Philosophicus Logicus)

Why Light travels in straight lines, Laws of reflection and Refraction, Quantum Mechanics, Newton’s Laws, Laws of everything!!

$$\text{Action} = S = \int_{t1}^{t2} (KE - PE)dt$$

$$\text{Least Action} = \frac{\partial S}{\partial t} = \frac{\partial}{\partial t}\left(\int_{t1}^{t2} (KE - PE)dt\right) = 0$$

$$\therefore KE - PE = 0 \therefore KE = PE$$

CONSERVATION OF ENERGY

Path dependent integrals – Calculus of Variations

Euler-Lagrange (E-L) equations

- **ENERGY IS CONSERVED**
  - **can neither be created or destroyed**
  - **can only be transformed from one form to another**

$$U_T = U_m + U_c + U_e + ... = Const$$

Where

U_T = Total Potential Energy
U_m = Mechanical Potential Energy
U_c = Chemical Potential Energy
U_e = Electrical Potential Energy
Etc……

$$\therefore \Delta U_T = \Delta U_m + \Delta U_c + \Delta U_e + ...? = 0$$

$$\therefore \Delta U_T = \Delta U_m + \Delta U_c + \Delta U_e + ...$$

$$Entropy(S) = 0$$

Entropy – double word

En – energy

Tropy – Trope (form)

Entropy – energy of change of form

- **HOW IS IT TRANSFORMED?**
  - **By doing work**

- **WHAT IS WORK?**
  - **Force X displacement**

$$W = \tilde{F} \bullet \tilde{\Delta}$$

Where

Work and energy are SCALARS
Force and displacement are VECTORS

$$\therefore \Delta U_T = W = \tilde{F} \bullet \tilde{\Delta}$$

Known as:
the WORK-ENERGY RELATIONSHIP

- **WHAT IS FORCE**
  - **A causal effect which changes the state of energy of a body**

- **Many types of force**
  - **Mechanical**

        - **Gravity**
        - **Motion (kinetic energy)**
    - **Electrical**
    - **Chemical**
    - **Same as types of energy**

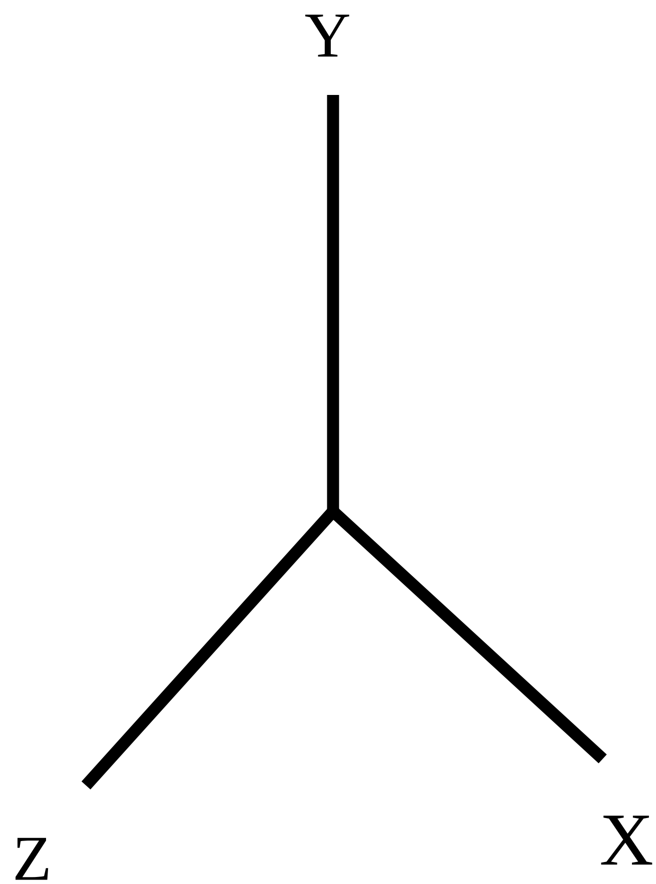
- **Different forces at different scales**
    - **Atomic forces**
    - **Real world forces**
    - **Cosmological forces**
    - **ENERGY ALWAYS CONSTANT**

- **HOW CAN WE MEASURE WORK/ENERGY?**
    - **Need to be able to define it**
    - **Need to be able to define force (not done until we do Dynamics)**

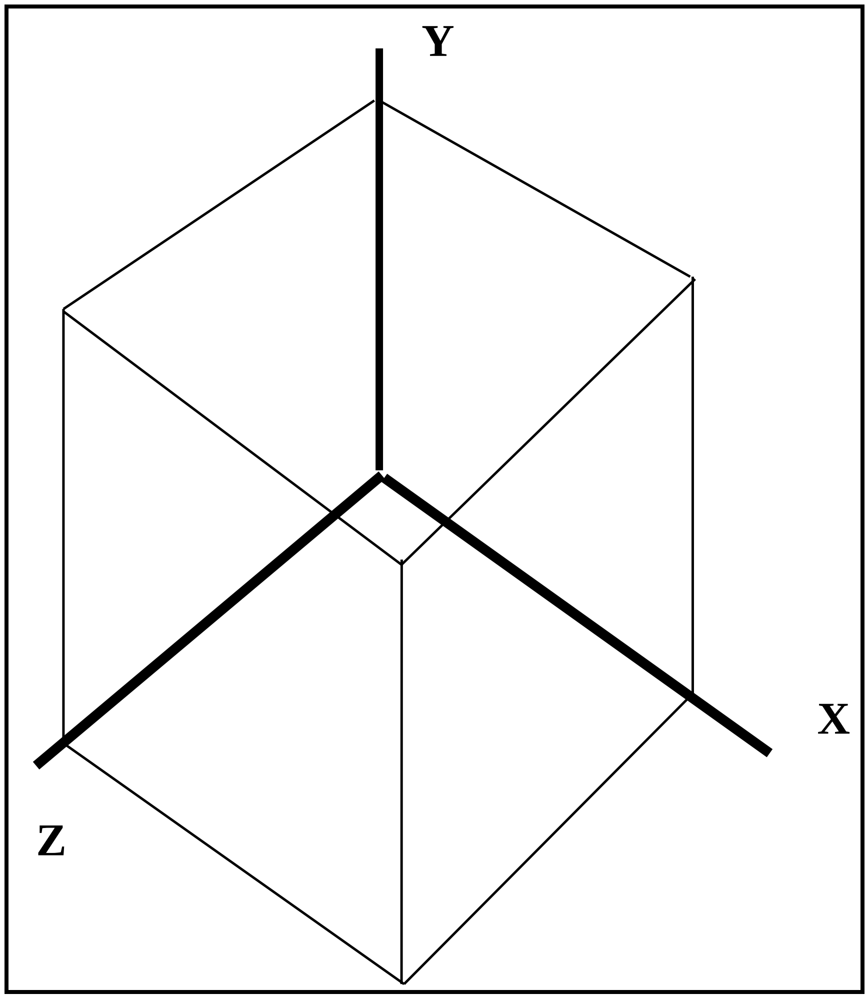
- Need to be able to define displacement
  - Need to be able to define position
- Need dimensions and units

- **DIMENSIONS/UNITS**
  - Mass - grams [kg]
  - Length - metre
  - Time – seconds
  - USA Imperial units!!!

- **HOW CAN WE DEFINE POSITION?**
  - Position means position in space
  - Need to be able to define space
  - 3D space
  - Cartesian co-ordinates (x,y,z)


Y
Z
X

3D SPACE



- **Point *'P'* is any point in space**
- **What does the line joining the origin (0,0,0) to the point *P* represent?**
- **THE POSITION VECTOR OF POINT *P***
- **A line (segment) which has both:**
    - **Magnitude and Direction**

LOOK AT VECTORS LATER

Problem Solving

In all cases, this can be broken down into the following steps:

1 Understand the problem or question (extract the relevant information)
2 Analyse the problem (decide on the mechanical principles/theory involved)
3 Define the problem in terms of equations or other mathematical means
4 Solve the equations
5 Conclude or explain the answer

The major difficulty is in deciding the appropriate theory to apply (and sometimes there may be alternatives, some easier than others). The choice is made on the basis of ease of use, and hence the need to practice the techniques of problem solving on a wide variety of different examples.

Conventional Idealizations

Frequently, the collection of things being studied are too complicated to be described by the available theory, or lead to equations that are too difficult to solve.
In these cases, it is often possible to obtain mechanical description by making certain idealizations or simplifications.

When problems are stated, there are certain conventional terms that are used to indicate different objects.

Particle : An object that has no physical size . Either the size is irrelevant to the problem, or the object is small compared with other objects being considered.

Light (adjective) means something is weightless, or it is negligibly small compared with other objects being considered.

Smooth (adjective) means there is no friction force between the objects in contact. Note the difference to standard English usage.

In other cases, the idealisation may be stated explicitly. For example, a wire may be said to be inextensible (that is, retains its length) or an object may be rigid (that is, its dimensions do not change).

Quantities We Shall Be Dealing With

Note in the following, some commonly used symbols are suggested, but others may be encountered from time to time.

Time: usually the duration of some event or action rather than a specific time. Units: seconds, s: Symbol: *t*

Displacement: change in position (measured from some datum). Units: metres, m: Symbol: **s**

Velocity: change in speed. Speed is the magnitude of velocity (that is, the direction is irrelevant) Units: $m\ s^{-1}$ Symbol: **v**

Acceleration: change in velocity. Units: $m\ s^{-2}$ Symbol: **a**

Mass quantity of matter. More usefully, it is the property of an object that determines its attraction to another object, or the property that determines how readily its acceleration can be changed by an applied force (see below).

These are referred to as the gravitational and inertial definitions of mass[1]. Units: kilogram, kg Symbol: *m*, *M* etc.

Note: Experiments have failed to find any difference between these two definitions. That is, however you choose to measure these quantities, if one object has twice the gravitational mass of another, it always has twice the inertial mass as well.

This is known as Einstein's "Theory of Equivalence" and led to the theory of General Relativity

Force: the effect that may cause a change in acceleration of an object. Units: newton, N Symbol: **F** sometimes **R – for reaction**

Weight: the force experienced by an object in a gravitational field. Units: Newton, N. Symbol: W or w.

Potential Energy: is the ability to do work. Units: Nm, symbol U.

Work: is the scalar product of force x displacement. Units Nm, symbol W (note be caeful it is the same symbol as weight). Work is the change in Potential Energy.

Work/Energy equation:

$$W = \Delta U$$

Where Δ (Delta, capital D in Greek)

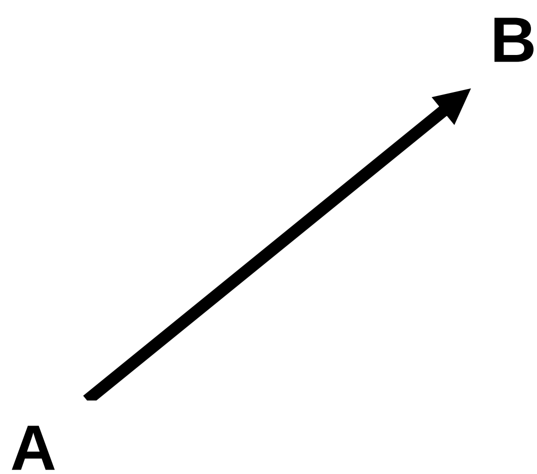
SCALARS AND VECTORS

Some of the above quantities can be characterised by a simple (single) number for example mass and energy. These quantities are scalars. With the others (for example displacement, velocity, acceleration and force) you need to know both a magnitude and a direction , otherwise you have insufficient information to work on (it is easy to think of examples to show this). Quantities that require more than one number to completely specify them are called vectors.

Mostly we shall be dealing with two–dimensional problems, and the vectors in these situations can be defined by two numbers. Occasionally we shall encounter three–dimensional vectors which are specified by three numbers.

Representation of Vectors

Since vectors are specified by a magnitude and direction, it is convenient to represent them in diagrams by a line (arrow) pointing in the appropriate direction, and with a length proportional to the vector's magnitude:



Take vector AB Obviously, with a simple line segment, there are two ways in which you could travel along the line, so normally the arrow head is added to indicate which direction is intended. The direction of travel is always from the first letter to the second i.e. from A to B.

Having introduced this concept, this can be used to show the of vector quantity represented.

Notation of Vectors

There are various ways of referring to vectors in written form. A common method is to label the arrow in a diagram, and use these with an arrow over the top of them:

$$\overrightarrow{AB}$$

means the vector from A to B

More usually, just write a symbol for the quantity i.e. **F**
In printed text, the symbol is **bold**, e.g. **a**,or **F**
In hand–written text, use a curly underline or overline.

Magnitude of a Vector

The magnitude of a vector **a** is written as **a** . Where there is no ambiguity, this may just be written as *a*.

Zero Vector

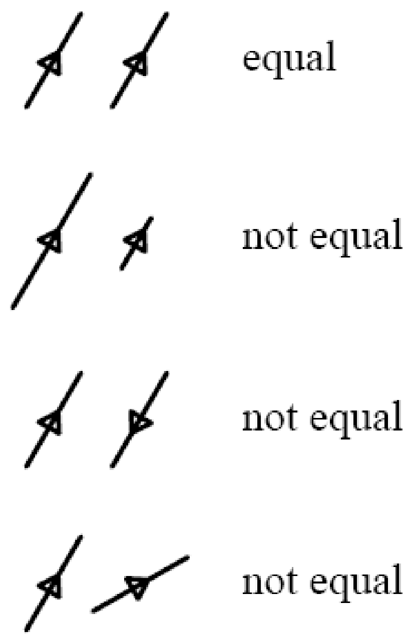
The zero vector is defined to be a vector with zero magnitude (its direction is irrelevant and is not defined).

Come back to this later.

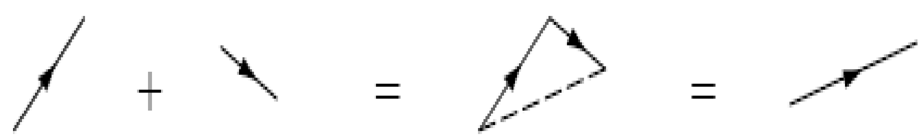
Mathematical Operations with Vectors

Equality

Two vectors are equal if their and are the same. For example:



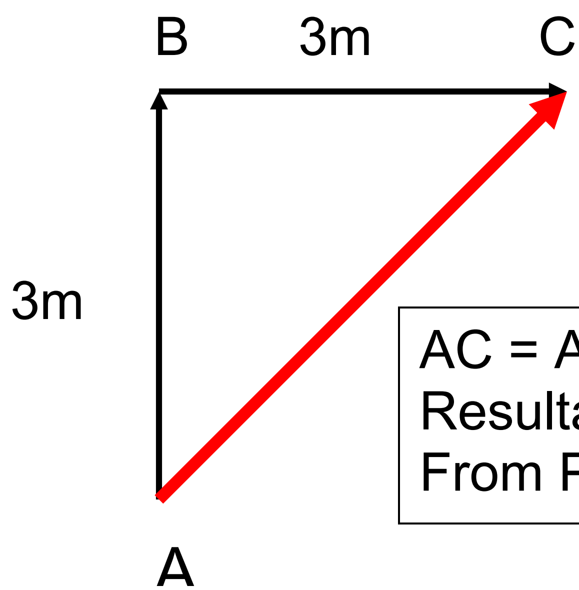
Addition of Vectors The result of adding two vectors (of the same type!) is the vector obtained as the vector sum of the triangle formed by the first two vectors, i.e



Note The figure illustrates the addition of two 2D vectors using a triangle as an example.

However, the mathematical rules of vector addition are the same whatever type of vector is used, and when it comes to understanding what is meant, for example, by two adding two vectors together, it is often useful to think in terms of the simplest type of vector.

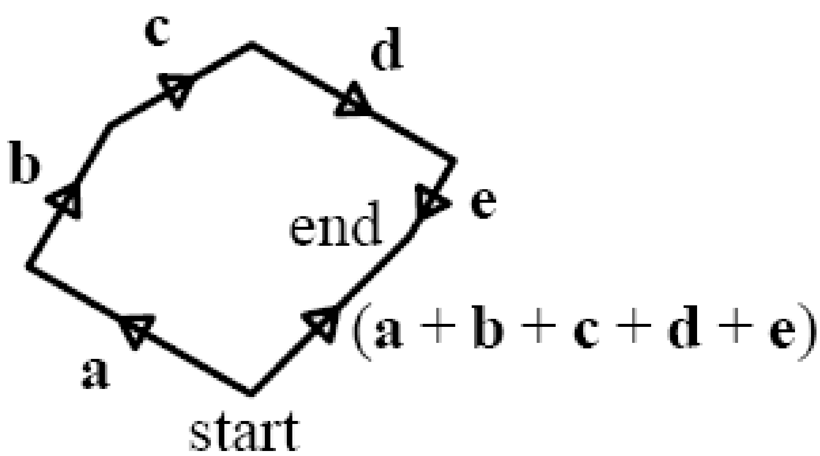
Thus if you have two displacements (one: move 3 metres in a northerly direction; two: move 3 metres in a an easterly direction) the effect of adding these two vectors is the result of adding one to the other, which is the same as moving a distance of 4.24 metres (approximately) in a north–easterly direction. Which is the resultant (the result of adding vectors)



AC = AB + BC =
Resultant = 4.24m
From Pythagoras

Adding vectors can be done by repeated application of the above rule. Generally, the sum of a number of vectors is obtained by drawing the vectors end to end to form an open Polygon. The sum is then the vector that closes the polygon (from a to e). Start from **a** and go **b to c** to **d** to **e** (**a** + **b** + **c** + **d** + **e**)

The sum or addition of a number of vectors is called the resultant of those vectors.



Note The most common mistake when constructing the resultant of two or more vectors is not ensuring that each vector starts at the end of the previous one, that is not having all the arrows pointing the same way round the triangle or polygon. (Observe, however, that the arrow on the resultant itself, (**a** + **b** + **c** + **d** + **e**) in the above diagram, points the other way round to all the others.)

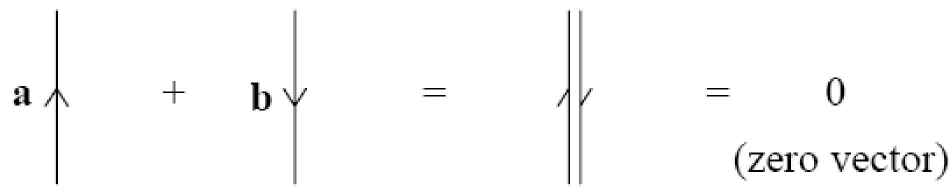
FUNDAMENTALS EXAMPLE 1

A boat is rowed across a river 20 m wide flowing with a speed of 0.5 ms $^{-1}$. If the boat travels through the water at a speed of 0.25ms^{-1}, and the bows the of the boat are always pointing directly at the opposite bank, how far downstream does the boat drift and what is the actual displacement of the boat?

FUNDAMENTALS EXAMPLE 2

A person runs 3 km due West, then 4 km due North and finally 5 km in direction South, 36.9° East. Show that they end up where they started from.

A Special Case



$$\mathbf{a} + \mathbf{b} = 0$$

so

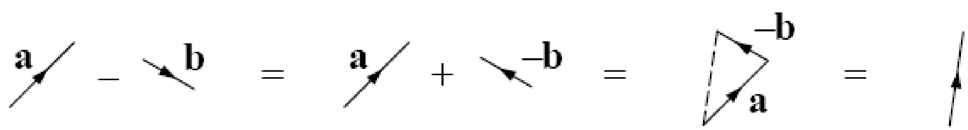
$$\mathbf{a} = -\mathbf{b}$$

Vector Subtraction

Subtraction of vectors can now be easily defined in terms of the addition of vectors:

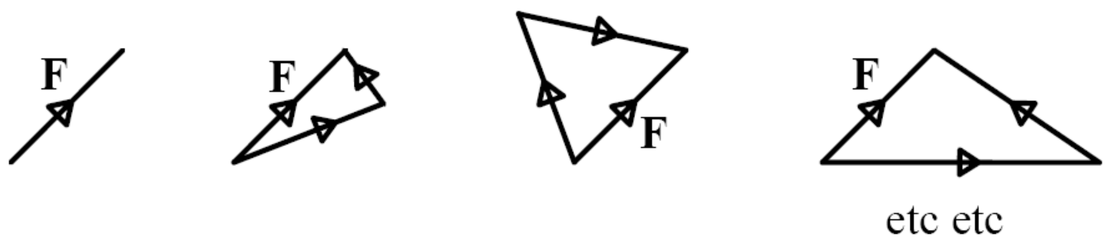
$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

So in this case, you would construct a vector summation diagram of **a** and the negative of **b** :



Components of Vectors

This is the reverse process to finding the resultant. Generally, any vector can be expressed as the sum of two vectors in an infinite number of ways:

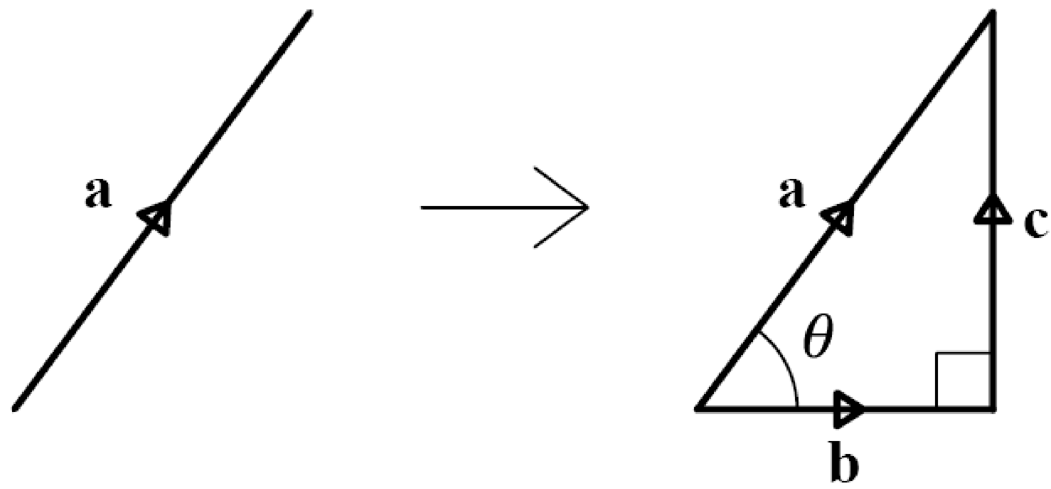


In principle a vector can be expressed as a sum of more than two other vectors, but two is the norm for two–dimensional problems, three is the norm for 3D problems etc.

It is often convenient to express a vector as the resultant of two mutually perpendicular (at 90°) vectors. These

are called the components f the vector. The choice of the components is not unique – it is usually determined by the particular problem under investigation.
Note A vector is then said to be resolved into its component form.

Having decided upon the two mutually perpendicular directions for the two components, it only remains to find the magnitudes of the two component vectors. Let θ be the angle between the vector (for example **a**) and one of the components :



The magnitudes of the **b**, **c** are then given by:

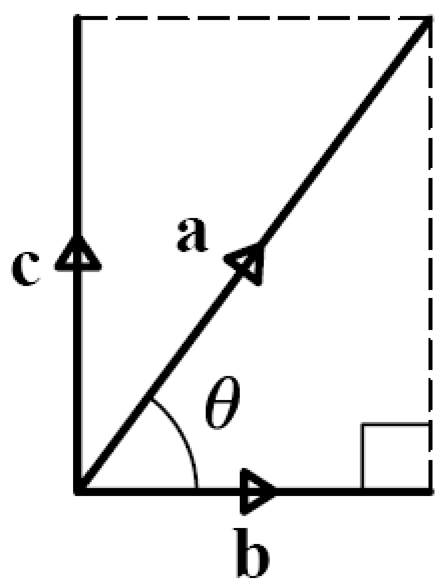
$$\| \mathbf{b} \| = \| \mathbf{a} \| \cos(\theta)$$

$$\| \mathbf{c} \| = \| \mathbf{a} \| \sin(\theta)$$

Where the symbol $\|?\|$ is known as the modulus or magnitude of the vector

Note this rule is remembered in the same way as the definitions of trigonometry are remembered: in the above diagram, **b** is adjacent to (next to) the chosen angle θ while **c** is opposite it.

Often, the vector and its two components are drawn with the origin as the datum:



If the components of a vector are parallel to the co–ordinate axes being used, it is possible to describe the vector by an equation linking its *x* and *y* components.

Components are useful because they can, to a certain extent, be treated separately.

For example, an alternative way of finding the resultant of a number of vectors is to resolve each into the same two directions, and to add (sum) the components separately. If required, the resultant of the summed components can then be found.

The advantage of this approach is that when summing the components, the vectors are all in the direction (or exactly opposite to it) so it is not necessary to draw a vector diagram, just to add up a list of quantities, as the following example shows.

FUNDAMENTALS EXAMPLE 3

Find the resultant of the following vectors:

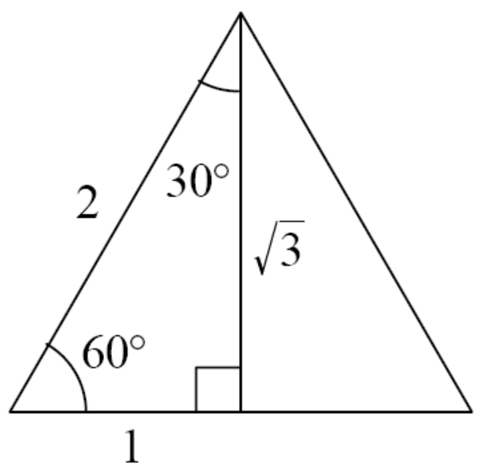
(a) magnitude 2 in the positive x direction
(b) magnitude 1 in the positive y direction
(c) magnitude 1 in a direction at 45° to the negative x and positive y directions
(d) magnitude 2 in a direction at –30° to the positive x direction.

Commonly Occurring Trigonometric Values

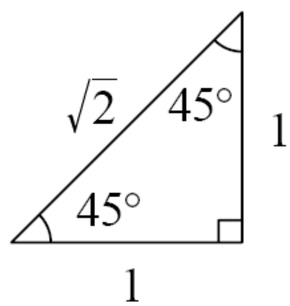
At this point, it is worthwhile listing certain trigonometric values that crop up frequently in vector problems (not just in mechanics, and not just this year). When they are encountered, it is usually a good idea to substitute the expression or value from the table, as this will often allow an apparently complicated formula to be simplified.

	sine	cosine	tangent
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

With frequent use you will find that you can remember these without difficulty, but to start with it is useful to memorise (be able to analyse!!!) the following triangles:



equilateral triangle
side length 2 (bisected)



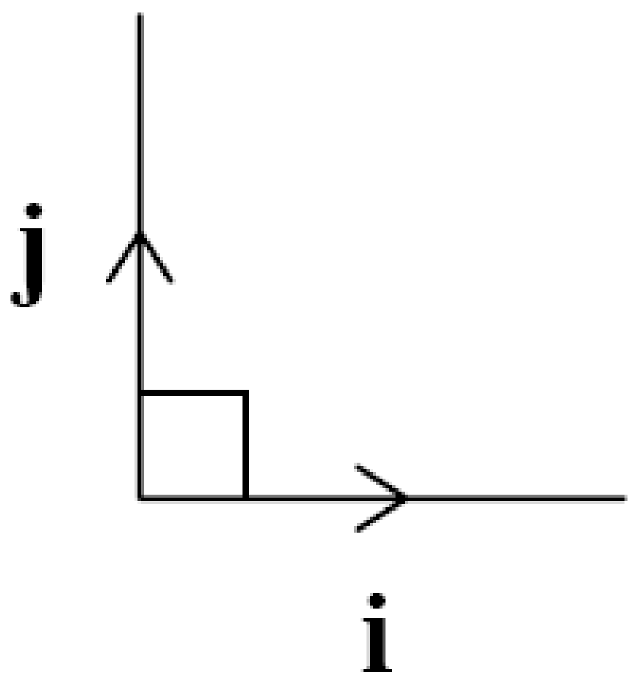
right–angled triangle
side length 1

These will allow you to obtain any of the values in the table very quickly (the lengths $\sqrt{2}$ and $\sqrt{3}$ do not have to be memorised as they can be determined easily using Pythagoras).

Cartesian Unit Vectors

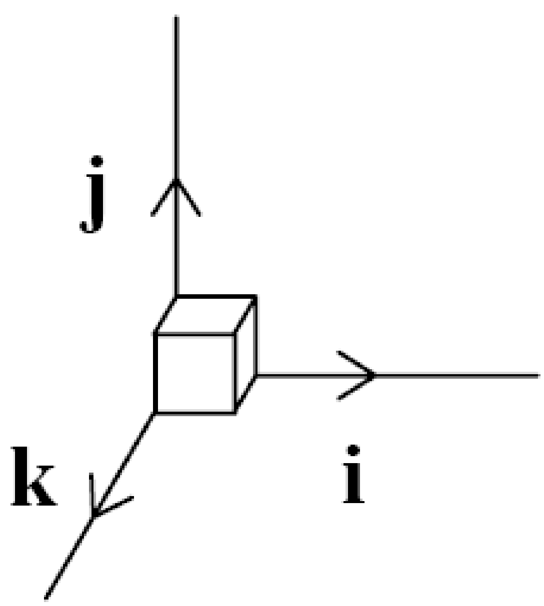
When resolving vectors into horizontal and vertical directions, it is often convenient to express the components as multiples of UNIT VECTORSin the two chosen directions.

In 2D problems, **i** and **j** are conventionally used to represent the unit vectors in the x and y directions respectively:



Note **i** and **j** do not have to be horizontal and vertical, but it is important that turning from direction **i** to direction **j** by the shortest route is a rotation of 90° (the same way you would choose *x* and *y* when constructing a graph). This is called a co-ordinate transformation.

In problems involving the direction perpendicular to both of these, **k** is also used. Again, the direction of **k** is chosen according to a RIGHT HAND RULE:



As a set of mutually–perpendicular direction is being used, **i** , **j** and **k** are called ORTHOGANOL UNIT VECTORS .

For example, in the previous class problem, vector **c** had components $1/\sqrt{2}$ in the negative *x* direction (and so minus) , and $1/\sqrt{2}$ in the positive *y* direction (and so plus). Its components are therefore

$$-\frac{1}{\sqrt{2}}\mathbf{i} \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\mathbf{j}$$

so we could write

$$\mathbf{c} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

Sometimes this might be written as

$$\left(-\frac{1}{\sqrt{2}}\ ,\ \frac{1}{\sqrt{2}}\ \right)$$

Which is analogous to writing (x,y) as (**i**,**j**)

A 3D vector might be 2**i** – **j** + 3**k** or (2, –1, 3) . So Fundamentals Example 3 could be done as follows:

$$\mathbf{a} = 2\,\mathbf{i}$$

$$\mathbf{b} = \mathbf{j}$$

$$\mathbf{c} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{d} = \sqrt{3}\,\mathbf{i} - \mathbf{j}$$

The resultant is

$$(2 - (1/\sqrt{2}) + \sqrt{3})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$$

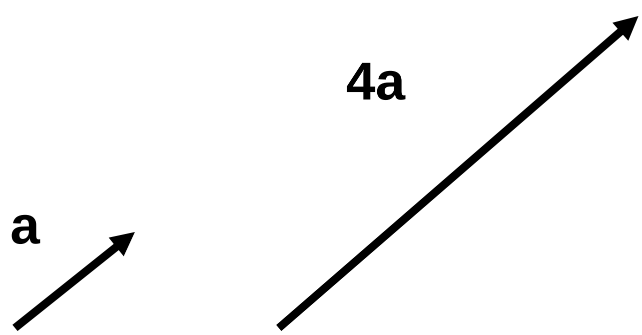
As before.

Expressing vectors in terms of Cartesian unit vectors is particularly useful when considering the multiplication of of vectors.

Multiplying Vectors

(i) Multiplication of a Vector by a Scalar

This is straightforward and we have already used the notion of multiplying by a number (scalar). In this case, the direction is unchanged, and the magnitiude is multiplied by the scalar number. So 4**a** is 4 times as long as **a** , but in the same direction.



When we say a vector has a component of – 3**i** , we mean it has a magnitude of 3 (3 times the length) and is in the direction opposite (negative) to **i** . If a vector is expressed in terms of components , just multiply all the components by the scalar. For example:

if $\mathbf{a} = \mathbf{i} - 3\,\mathbf{j}$

then $4\,\mathbf{a} = 4\,\mathbf{i} - 12\,\mathbf{j}$

(i) Multiplication of a Vector by Another Vector

We have seen that adding vectors together sums their (force for example) or is equivalent to applying one vector after the other (as with displacements).

In certain situations, vectors combine to produce an effect which depends upon the sign of their magnitudes, and the angle between the vectors.

In fact, there are two ways they might be combined as a product, depending upon whether the combined effect is greatest when the two vectors are parallel, or when they are at perpendicular to each other. The type of quantity obtained as the product is quite different in these two cases.

Note: When we were adding vectors together, it was important that they are vectors of the same type (all forces, all displacements etc). When multiplying vectors together, it is not necessary for the two vectors to be of the same type. In fact, in the mechanics examples we shall come across, the two vectors will be of different types.

(a) Scalar Product

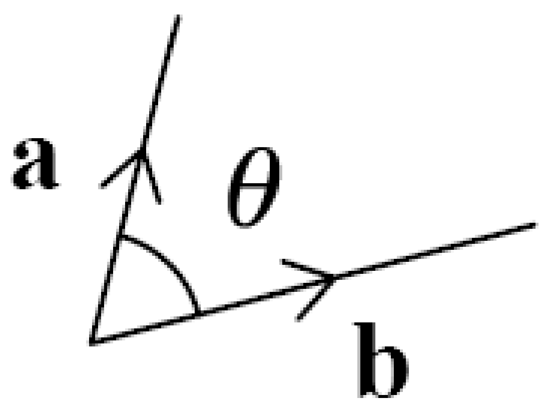
As its name suggests, the result in this case is a number – a scalar.

This type multiplication operation is normally denoted by placing a dot between the two vectors, hence it's alternative name is a dot product for example: **a** . **b**

The scalar product arises when the effect of multiplying the two vectors is a maximum when these are parallel (assuming their sizes are of the same sign). Formally, it is defined as follows:

$$\mathbf{a} \cdot \mathbf{b} = \| \mathbf{a} \| \text{ x } \| \mathbf{b} \| \cos(\theta)$$

where x denotes multiplying the modulii (magnitudes) together and θ is the angle between the two vectors:



So if **a** is parallel to **b** :

$$\mathbf{a} \cdot \mathbf{b} = \| \mathbf{a} \| \times \| \mathbf{b} \|$$

because cos(0) = 1 , but if **a** is perpendicular to **b** :

$$\mathbf{a} \cdot \mathbf{b} = 0$$

because cos(90) = 0

Note The scalar product is commutative (the order of multiplication is irrelevant), that is

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \,.$$

This follows immediately from the definition.

Scalar products of UNIT VECTORS are particularly simple:

$$\mathbf{i} \cdot \mathbf{i} = \|\mathbf{i}\| \text{ x } \|\mathbf{i}\| \cos(0) = 1 \text{ x } 1 \text{ x } 1 = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \|\mathbf{i}\| \text{ x } \|\mathbf{j}\| \cos(90) = 1 \text{ x } 1 \text{ x } 0 = 0$$

So:

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

($\mathbf{k} \cdot \mathbf{j}$ etc follow from commutative law).

This leads to a very simple way of scalr multiplication when vectors are expressed in terms of Cartesian unit vectors:

if $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$

$\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$

then

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= a_1 b_1 \, \mathbf{i}\cdot\mathbf{i} + a_1 b_2 \, \mathbf{i}\cdot\mathbf{j} + a_1 b_3 \, \mathbf{i}\cdot\mathbf{k} \\ &\quad + a_2 b_1 \, \mathbf{j}\cdot\mathbf{i} + a_2 b_2 \, \mathbf{j}\cdot\mathbf{j} + a_2 b_3 \, \mathbf{j}\cdot\mathbf{k} \\ &\quad + a_3 b_1 \, \mathbf{k}\cdot\mathbf{i} + a_3 b_2 \, \mathbf{k}\cdot\mathbf{j} + a_3 b_3 \, \mathbf{k}\cdot\mathbf{k} \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3\end{aligned}$$

using the scalar product definitions of UNIT VECTORS given above.

This simple rule for calculating the scalar product ("multiply corresponding coefficients, and add these up") is very widely used and applies equally well to 3D as it does to 2D.

As a corollary, if the scalar product of two vectors is zero, they must be perpendicular .

FUNDAMENTALS EXAMPLE 4

Find the scalar product of the two vectors:

$3\,\mathbf{i} + 2\,\mathbf{j}$

and

$-\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$

Solution:

$$\text{scalar product} = 3 \text{ x } (-1) + 2 \text{ x } 3 + 0 \text{ x } 2$$

$$= -3 + 6 = 3$$

Scalar products are also useful in calculating the resultant of two vectors, assuming these are defined in component form. First, you calculate the magnitudes of the resultant from their components as explained above, which is relatively easy. Next, you find the magnitude of each vector using Pythagoras (which works just as well in 3 dimensions as it does in 2!) that is:

$$\| \mathbf{a} \| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Finally, you calculate the magnitude of the angle between the vectors using the definition of the scalar product:

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{\| \mathbf{a} \| \times \| \mathbf{b} \|}$$

We shall encounter scalar products when we come to the concept of later on in the module.

(b) Vector Product

We will not cover this in this module, you will encounter it later on in the First Year.