# KINEMATICS II

## PROJECTILES

A projectile is an object that is given an initial velocity, and is then acted on **only by gravity** .

We have already studied a special case of projectile motion in Unit 5: Kinematics I when we looked at objects moving in a **parabola** under the action of **gravity** . As usual at this stage, we simplify the problem by **ignoring air resistance**.

The motion of projectiles is determined from two basic
principles:

1. There is no force acting in the horizontal direction, so there is no horizontal acceleration. As a consequence, the horizontal component of the velocity of the projectile is **constant**, and will always be equal to whatever it was to start with.

2. The vertical component of velocity is subject to the force of gravity of *g* downwards. As this is generally a constant value[1] the equations of motion derived here can be used to describe the movement in the vertical direction.

We therefore treat the horizontal and vertical components of the motion separately. In this way, everything we need to know about the motion of the projectile can be derived once we know the initial condition (velocity).

Conventionally, these are specified As a velocity and an angle.

[1]providing the change in height is not too large – another assumption

## KINEMATICS EXAMPLE 8

A stone is projected with a velocity of 10 m $s^{-1}$ at an angle of 60° to the horizontal. Find:

- (i) the velocity after 0.5 seconds
- (ii) the height of the stone above its starting point when it has travelled a distance of 1 metre
- (iii) the time taken for it to reach a position with the same height as the starting point

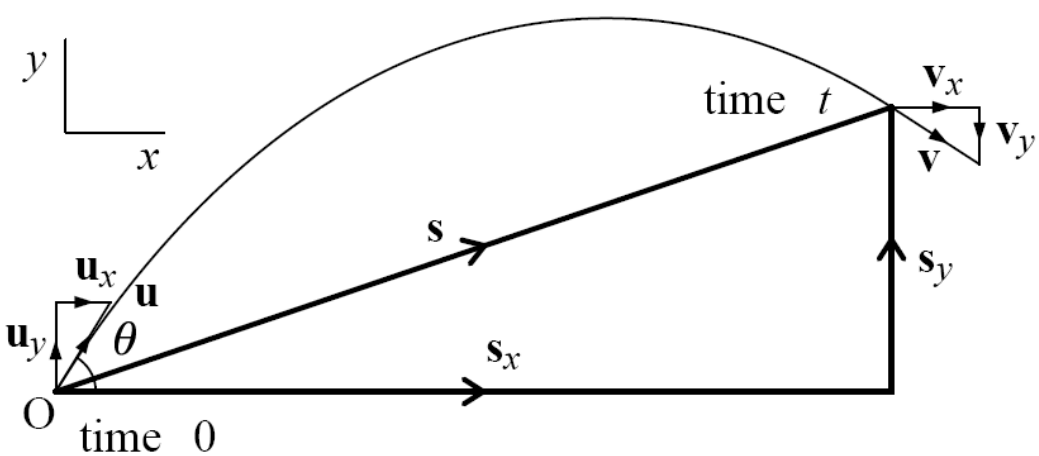
The previous example illustrates the range of calculations we might want to perform for projectile motion. These calculations will be pretty much the same for any projectile, but with different numbers. It is tedious to start the calculation from first principles each time, so as is usual in these situations, it is convenient to obtain a set of standard equations that we can apply to any example of projectile motion.

# General Equations of Motion of a Projectile

Let the initial velocity (a vector) of the projectile be **u** acting at an angle of θ to the horizontal.

Let **v** be the instantaneous velocity at some **time** *t* , when the displacement is **s** measured from **origin**.

Let the *x* axis be horizontal, and the *y* axis vertical with + *y* measured upwards. *x* and *y* subscripts are used to denote the horizontal and vertical components of the various quantities.



Initially (at time 0):
horizontal velocity $u_x = u \cos\theta$
vertical velocity $u_y = u \sin\theta$

At time $t$ :
Horizontal Velocity

$$v_x = u_x = u \cos\theta \text{ (constant)}$$

Vertical Velocity

$$v_y = u_y - gt \text{ (g =constant acceleration)}$$

$$v_y = u \sin\theta - gt$$

Horizontal Displacement

$$s_x = ut \cos\theta$$

Vertical Displacement

$$s_y = u_y t - 1/2gt^2$$

$$s_y = ut \sin\theta - 1/2gt^2$$

# Equation of Path or Trajectory

Combine the equations for horizontal and vertical displacement by eliminating *t* :

$$
\begin{aligned}
t &= \frac{s_x}{u\cos\theta} \quad \text{(from horizontal displacement)} \\
s_y &= u\frac{s_x}{u\cos\theta}\sin\theta - \frac{1}{2}g\left(\frac{s_x}{u\cos\theta}\right)^2 \\
s_y &= s_x\tan\theta - \frac{g}{2u^2\cos^2\theta}s_x^2
\end{aligned}
$$

If we choose to place the origin of the axes at the starting point, the displacement **s** is the same as the distance **x** , with components $S_x$ and $S_y$ Hence the vertical **height** can be expressed as:

$$
y = \tan\theta \; x - \frac{g}{2u^2\cos^2\theta}x^2
$$

Note that this expression for *y* is a quadratic (contains squared terms) in *x*, and therefore defines a parabola.

**Time for Maximum Height**

Assuming the height is positive (above the horizontal) this is the time for the projectile to **reach the maximum height** in its trajectory. This will occur when the vertical component of its velocity becomes (instantaneously) zero (the horizontal velocity of course is constant throughout this). Use the previous equation for $v_y$ to find the time $t_h$ for this to happen (measured from the instant of projection):

$$
\begin{aligned}
v_y &= u \sin\theta - g t_h = 0 \qquad \text{(for max height)} \\
t_h &= \frac{u \sin\theta}{g}
\end{aligned}
$$

## Maximum Height

Substituting this expression for *th* into the equation for the vertical component of the displacement $s_y$ gives the maximum height reached during the projectile motion (again, assuming θ is known):

$$h = u\frac{u\sin\theta}{g}\sin\theta - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$
$$h = \frac{(u\sin\theta)^2}{2g}$$

## Horizontal Range

This is the distance travelled in the horizontal direction when the projectile once again returns to its **vertical datum** (i.e. vertical displacement is again zero ). We could use the equation for the path to obtain this, but it is more convenient to once again use the expression derived for the **vertical component of displacement** $s_y$ :

$$s_y = ut\sin\theta - \frac{1}{2}gt^2 = 0$$

or

$$(2u\sin\theta - gt)t = 0$$

This has two solutions:

$$t = 0 \quad \text{and} \quad t = \frac{2u\sin\theta}{g}$$

The first of these is the trivial solution for the start of the projectile motion; the second is the time for the projectile to **Return** to its starting height, substituting this into the expression for horizontal displacement gives:

$$\begin{aligned} s_x &= ut\cos\theta \\ &= u\frac{2u\sin\theta}{g}\cos\theta \\ &= \frac{2u^2\sin\theta\cos\theta}{g} \end{aligned}$$

Making use of the trigonometric identity

$$\sin 2\theta = 2\sin\theta\cos\theta$$

gives the following expression for horizontal range *r* :

$$r = \frac{u^2 \sin 2\theta}{g}$$

**Time of Flight**

This is the time taken for the projectile to **return to** its starting height (i.e. time to travel the maximum displacement). We have therefore already obtained this expression (see above):

$$t_r = \frac{2u\sin\theta}{g}$$

**Maximum Horizontal Range**

It is often important to know the maximum distance an object can be projected. Specifically, given that the speed of projection *u* is a constant, at what angle must the object be projected in order to give it the greatest possible horizontal distance, and what is the value of this angle?

From the expression for the horizontal range:

$$r = \frac{u^2 \sin 2\theta}{g}$$

If *u* is fixed, this will be a maximum when sin 2θ has its maximum value, i.e. 1. Therefore, for maximum range:

$$\begin{aligned} \sin 2\theta &= 1 \\ 2\theta &= 90^o \\ \theta &= 45^o \end{aligned}$$

where we choose, for obvious reasons, the solution for 2θ lying in the range $0^o$

. to 180$^o$. The range of the projectile will therefore be a maximum when the angle of projection is at 45$^o$. to the horizontal. Substituting the value sin 2θ = 1 into the expression for the horizontal range gives the maximum value of this to be:

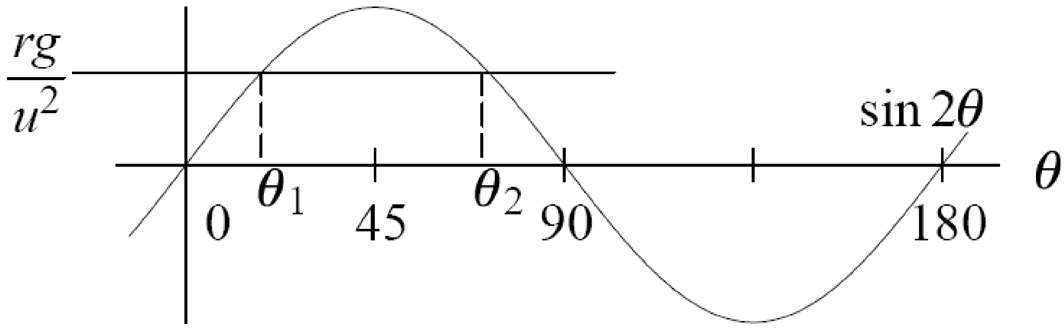
$$r_{\max} = \frac{u^2}{g}$$

## Angle of Projection for a Given Range

Given a distance $r$ with $r \leq r_{\max}$, for some fixed initial velocity *u* , at what angle θ to the horizontal must the object be projected to have a displacement equal to *r* ? Again, we use the expression for horizontal range:

$$r = \frac{u^2 \sin 2\theta}{g}$$
$$\sin 2\theta = \frac{rg}{u^2}$$

There are two values of θ between 0° and 90° that will satisfy this equation:



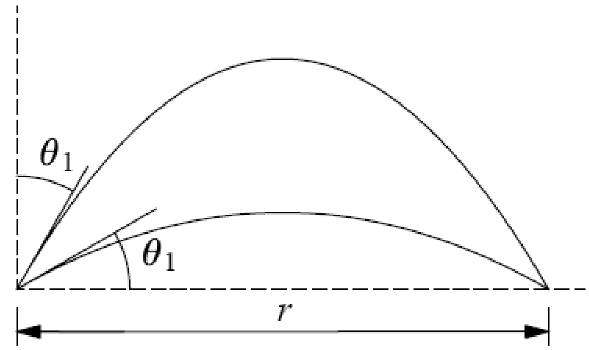

Analytically:

$$2\theta = \arcsin\frac{rg}{u^2} \qquad \text{or} \qquad 2\theta = \pi - \arcsin\frac{rg}{u^2}$$

so, in degrees:

$$\theta_1 = \frac{1}{2}\arcsin\frac{rg}{u^2} \qquad \text{or} \qquad \theta_2 = 90 - \theta_1$$

Thus there are in theory **two** possible angles of projection that will provide the required range, one at $\theta_1$ to the horizontal direction, one at $\theta_1$ to the vertical direction. These lead to two different parabolas, with different paths etc.



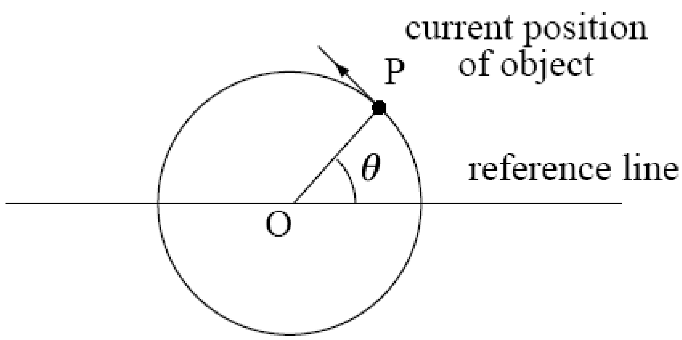

KINEMATICS EXAMPLE 9

A ball is thrown with an initial speed of 10 $ms^{-1}$ from a point 2 m above the ground so that it just clears a fence 4 m high situated a horizontal distance of 5 m from the ball's starting point. Show that there are two possible angles the initial velocity can make with the horizontal, and find their values.

Although it is useful to memorise the standard equations of motion for projectiles, they can always be derived from the components when required. Indeed, you might be asked in an examination question to do just that.

# CIRCULAR MOTION

Circular motion is frequently met in mechanical systems, so it is important to study the behaviour of objects moving in a circle, or the motion of points on bodies that are rotating .
For objects moving in a circular path, an alternative (and sometimes more convenient) description of their motion can be obtained in terms of angular measures of displacement, velocity and acceleration.
For example, the angular displacement of a point at P with respect to an origin, centre O is simply the angle between the radius, and some reference or datum:

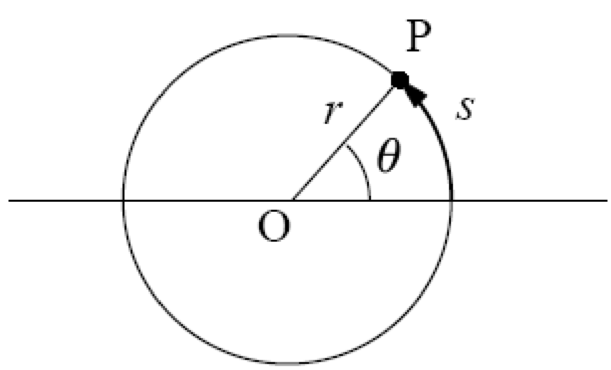


$\theta$ is measured in an anticlockwise manner. The choice of the reference line is, of course, arbitrary, as is the choice of the reference position from which the rotation is measured.

The relationship between the angle $\theta$ and angular displacement is obtained very simply:

$$s = r\theta$$

where $r$ is the radius of the circular path and $s$ is the angular or radial displacement to the same reference line.



If the object at P is moving, $\theta$ will increase and so we need to define the **change** of $\theta$ with respect to **time**. This is called the **angular velocity** of the object and is usually denoted by $\omega$

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}$$

(**Note** the use of the dot over the variable to denote differentiation or rate of change of the variable with respect to **time**)

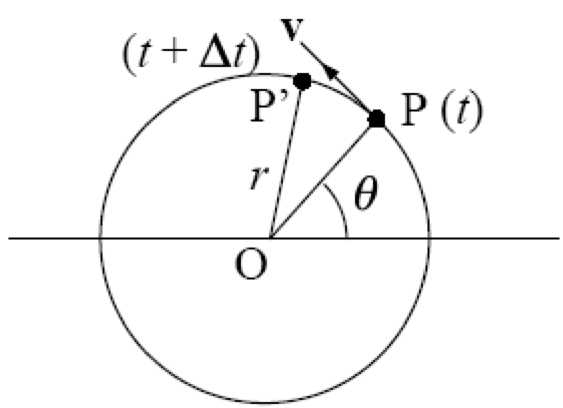
Similarly, $\omega$ may vary with time, and so we can define the angular acceleration α of the object:

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \dot{\omega} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \ddot{\theta}$$

***Note*** we have actually defined the dirction of the $\vec{\theta}, \vec{\omega}$ and $\vec{\alpha}$ vectors, the directions of which are given by the axis of rotation.

## Relationship Between Linear and Angular Velocity

Let the radius of the circular path be *r* . Let the object be at P (angle $\theta$) at time *t* and have a linear velocity **v** at that time (i.e. a tangential velocity of *v* around the circle). Suppose a short time $\Delta$t later, the object is at P':




If $\Delta t$ is very small:

$$v \simeq \frac{\text{distance round circle from P to P'}}{\Delta t}$$

But **the displacement from P to P' is**:

$$\text{PP'} = r\ \text{P'}\hat{\text{O}}\text{P}$$

where $\text{P'}\hat{\text{O}}\text{P}$ is the increase in the angle $\theta$ in time $\Delta t$ , [put in diagram]

i.e. $\Delta\theta = \omega\Delta t$ so:

$$v = \frac{r\omega\Delta t}{\Delta t}$$
$$v = r\omega$$

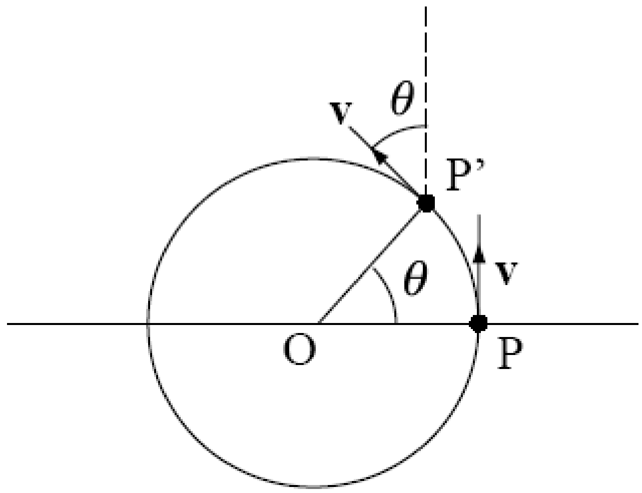
***Note*** we previously obtained the relationship $s = r\theta$ between linear and angular displacement. Do you think there might be a similar relationship between linear and angular acceleration? i.e. $a = r\alpha$

# Circular Motion with Constant Speed

In this special case, we have $v = r\omega$, and hence $\omega = v/r$. By definition therefore α is $\frac{d\varpi}{dt}$. However, linear acceleration is the rate of change of the **tangential velocity**, and for circular motion the dirction of the velocity vector is constantly changing, even if the size of the velocity (speed) is constant.
The linear acceleration **a** is therefore $\frac{dv}{dt}$, thus disproving the assertion that
$a = r\alpha$

Having established that an object travelling in a circle at constant speed is accelerating, we need to find out what this acceleration is. As before, consider an object initially at P at time 0 travelling at constant speed *v* so that it is at position P' at some time *t* later.

For convenience, let the angular displacement at P be **zero**:




Resolve the velocity vectors at P and P' (at θ) to the direction OP (which we shall call the **Radial** direction) and tangentially to OP (called the **Tangential** direction).

At time 0:

radial component of velocity: 0
tangential component of velocity: *v*

At time *t* :
radial component of velocity: $-v\sin\theta$
tangential component of velocity: $v\cos\theta$

So the **accelerations** in these two directions are:

average radial acceleration: $\dfrac{-v\sin\theta - 0}{t} = \dfrac{-v\sin\theta}{t}$

average tangential acceleration: $\dfrac{v\cos\theta - v}{t} = \dfrac{v(cos\theta - 1)}{t}$

But we want the **instantaneous acceleration** at P (time zero) so we must take the **limit** as $t \to 0$ of the above expressions.

Because $\omega$ is the angular velocity then by definition:

$$\omega = \frac{\theta}{t} \qquad or \qquad t = \frac{\theta}{\omega}$$

so:

$$\frac{-v\sin\theta}{t} = -\frac{v\omega\sin\theta}{\theta}$$
$$\frac{v(cos\theta-1)}{t} = \frac{v\omega(cos\theta-1)}{\theta}$$

As $t \rightarrow 0$, $\theta \rightarrow 0$ and:

$$\frac{\sin\theta}{\theta} \rightarrow 1$$
$$\frac{\cos\theta-1}{\theta} \rightarrow 0$$

Hence:
radial component of acceleration = – $v\omega$

tangential component of acceleration = 0

The **radial component** of acceleration is normally expressed in one of two alternative forms by using the relationship $v = r\omega$ to replace either $v$ or $\omega$ i.e:

or:

$$a_r = -\frac{v^2}{r}$$

$$a_r = -r\omega^2$$

To summarise, if an object is travelling in a circle of radius $r$ at constant linear or tangential velocity $v$ (or constant angular velocity $\omega$) its radial acceleration is $\frac{v^2}{r}$ (or $r\omega^2$) directed towards the centre of the circle (used to be called the Centripetal acceleration)

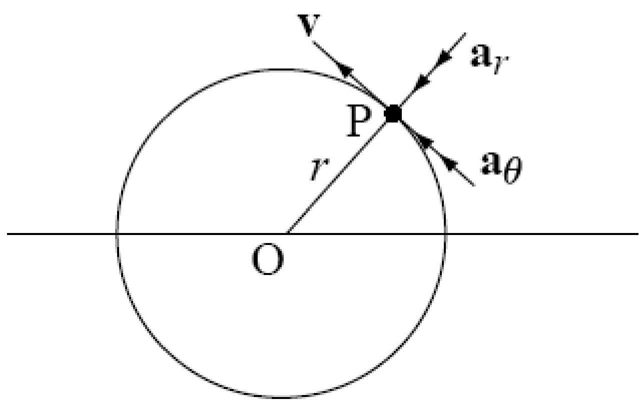
***Note*** the last remark means that the linear acceleration is not only changing, it is not zero either.

# KINEMATICS EXAMPLE 10

A car of mass 1 tonne travels at a constant speed of 90 km $h^{-1}$ round a bend of radius 100 m in a level road. What is the reaction of the road on the car while it is traversing the bend?

# Circular Motion with Non–constant Speed

Let the object at P be moving with varaible speed *v* around a circle of radius *r* . Let the **linear acceleration a** at P have components $\mathbf{a}_r$ **directed towards the centre of the circle** O of the circle, and $\mathbf{a}_\theta$ **tangential** the circle in the anticlockwise direction.




Consider $\mathbf{a}r$ . At the instant under consideration, the radial acceleration cannot be affected by any change in the speed of the motion at **$90^o$** to the radius (i.e. tangentially). The magnitude of $\mathbf{a}_r$ is therefore the same as for the constant velocity motion, except *v* is now the instantaneous speed:

$$\mathbf{a}_r = \frac{v^2}{r} = r\omega^2$$

Now consider $\mathbf{a}_\theta$. At the instant under consideration, the acceleration in the tangential direction will be the rate of change of the **velcoity** in that direction. But the velocity vector is a **tangent** to the circle, so $\mathbf{a}_\theta$ is just the rate of change of speed *v* .

But $v = r\omega$ so:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \dot{v} = r\frac{\mathrm{d}\omega}{\mathrm{d}t} \qquad \text{(because r is constant)}$$
$$= r\alpha$$

Hence:

$$\mathbf{a}_\theta = \dot{v} = r\dot{\omega} = r\alpha$$

# KINEMATICS EXAMPLE 11

A particle P of mass $m$ is attached to one end of a light inextensible string of length $r$, the other end of which is attached to a fixed point O. P is projected horizontally from its lowest position so that its highest position during its subsequent motion is just level with O. Find the tension in the string and the rate of change of speed when OP makes an angle of 60° to the vertical.