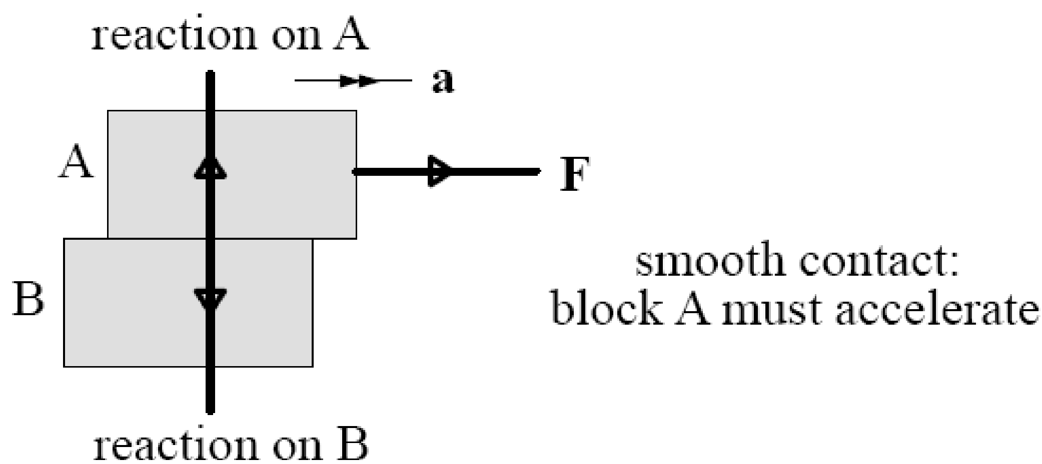
STATICS II

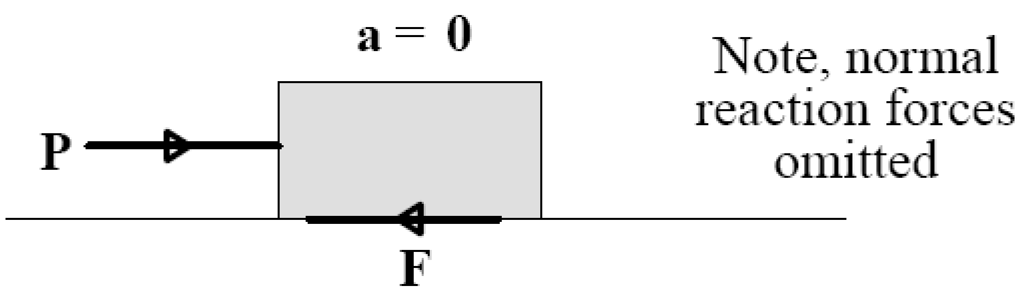
FRICTION

So far, we have mainly assumed that when bodies are in contact, there is no restraint acting to prevent slipping (i.e. we have assumed smooth surfaces). This means that the only force that can act at the interface between the two bodies is the **normal force** between them.



We shall now turn to the more usual case where relative sliding between the bodies is subject to resistance. This is called COULOMB FRICTION (or, less precisely, we say the **bodies are "rough"**).

As an example, imagine trying to push a heavy box over a rough floor. From experience, it is known that you have to apply a certain force P which is larger than the friction force F. Consider the situation where you are pushing with a smaller force than this:



But since the box is not accelerating, it must be in static equilibrium and there must therefore be a **friction force opposing** the applied force **P** (and **equal to P**). The only way this force can act on the box is due to contact with the floor. Therefore there is a **friction force** acting on the base of the box **F** such that **P** + **F** = **0**. This implies that **F** is parallel to the **applied force P** (or tangential to the surface).

This leads to the first statement we can make about friction:

1. When there is no relative sliding between two rough surfaces, the frictional force acting on each surface is in the opposite direction to the potential direction of relative movement of that surface.

By potential direction, we mean the direction the surface would move in if there was no friction.

As a general guideline:

Friction acts to prevent motion!

The second thing we can say about friction follows from the result **P** + **F** = **0** that we obtained above:

2. When there is no relative sliding between two rough surfaces, the magnitude of the frictional force acting on each surface is **equal and opposite.**

.

This may sound like a truism, but it is important because it means that for static contact the frictional force can vary in magnitude. The only way of determining its size is by examining all the other forces acting on the body and using the principle of equilibrium.

Statement 2 implies that the greater the force applied in order to move a body, the greater is the friction.

.

However, there is a **limit** to the size of the frictional force, and if a force

greater than the friction force is applied, the body will move (in fact accelerate, as the resultant force will be greater than zero).

Experiments show that for the sort of frictional contact found in mechanical systems (Coulomb friction) the maximum value of frictional force is equal to the normal force acting at the contact surfaces. This leads to the third statement about friction:

3. There is a limit to the frictional force that can act between two rough surfaces. The limiting value is found to be μN, where μ is the Coefficient of Friction, and N is the magnitude of the Normal Force between the surfaces.

For convenience, we can think of 3 different types (or states) of rough contact:

1. No relative movement between the bodies in contact: the frictional force will be **less than** the resultant of all the

other forces acting on one of the bodies in a direction which is parallel to the contact surface (i.e. static equilibrium).

2. Bodies just on the point of sliding: the frictional force will be **equal** to the resultant of all the other forces acting on one of the bodies in a direction which is parallel to the contact surface (because the body is still static) but also $F = \mu N$. This is an example of limiting friction**.**

3. Bodies sliding relative to one another: the size of the frictional force is given by $F = \mu N$ and the direction is opposite to the direction of motion of the body on which it acts.

Note in case 3, the frictional force may be equal and opposite to the other forces acting (body is moving with constant velocity) or
less than these (body is static).

STATICS EXAMPLE 8

A crate of mass 25 kg is just on the point of moving across a rough floor when the tension in a rope attached to the crate, and making an angle of 45^o. to the horizontal, is 150 N. What is the coefficient of friction between the crate and the floor?

Notes

1. Frictional force is assumed to be independent of the (apparent) **area of contact (Amonoton's Law)**.
2. We have assumed that $F = \mu N$ for the friction force and limiting friction conditions. In some situations, the frictional force when movement has been established is less than the value required to initiate movement.
It sometimes useful therefore to define different coefficients of friction for static and kinematic situations.
However, we shall not be doing this.
3. Inclusion of friction in a mechanical system dissipates energy (as heat), only its magnitude (maybe to the extent of preventing motion entirely).
4. The terms rough and smooth as used in the context of frictional problems are not meant as descriptions of the texture or appearance of the contact surfaces: two highly polished and clean surfaces

in contact can have a **coefficient of friction of unity.** .

Problem Solving

1. Frictional forces can be incorporated into statics and dynamics equations just like any other forces. The main difference however is that you have to get the magnitude and direction of the friction force correct, or the equations will not give the right answer (with most other types of **force**, if you guess the wrong direction, you just obtain a negative value at the end). The reason why friction is different is that it acts to **dissipate energy** (potential or actual) and so reversing the direction of the frictional force is **not permissible**
2. All is not lost however, because you can always obtain the direction of potential motion of the various parts of

the mechanical system by first assuming zero friction .

Note 2 above ensures that when **friction** is put back in, the directions of motion will not alter (although actual movement may become potential movement).

3. Another point to be careful about is that $F = \mu N$ can only be used for **sliding/slipping** (or just on the point of sliding/slipping). So look out for phrases like 'just on the point of slipping' or 'just sufficient to prevent movement' etc. In general you can rely on the fact that $\mathbf{F} \leq \mu \mathbf{N}$.

4. One final word of warning: do not fall into the trap of assuming that $N = mg$ (normal reaction equals the weight of the object). Sometimes it is, but often it is not.

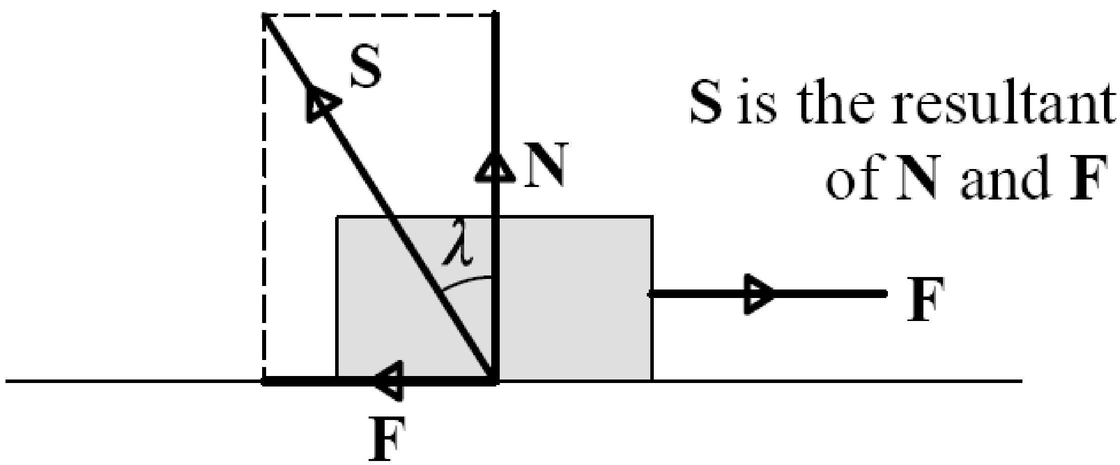
STATICS EXAMPLE 9

A particle is placed on a plane surface which can be inclined at different angles to the horizontal. If the coefficient of friction between the particle and the surface is μ, find the angle of inclination of the surface when the particle is just about to slip down.

Resultant Contact Force

Whereas for smooth contact the only force acting at the contact surfaces is the normal reaction, with rough contact, there is also a frictional force acting

:



The resultant contact force **S** acts at an angle γ to the direction of the normal contact force **N**. By definition:

$$\tan\lambda = \frac{F}{N}$$

Angle of Friction

For limiting friction, $F = \mu N$, so the above equation becomes:

$$\tan\lambda = \frac{\mu N}{N} = \mu$$

so:

$$\begin{aligned} \tan\lambda &= \mu \\ \lambda &= \arctan\mu \end{aligned}$$

. γ is the angle of friction, i.e. the angle that the resultant force makes with the normal to the contact surfaces.

γ is useful, because once γ is known, the **magnitude** of the resultant contact force can be found immediately, thus allowing one force to be used in place of two at the contact surface

.

STATICS EXAMPLE 10

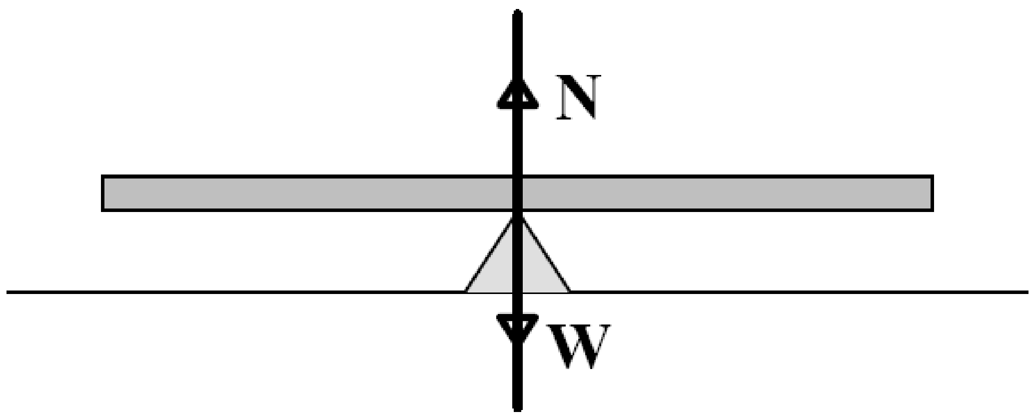
A uniform rod AB of length 2*a* and weight *W* is inclined at an angle of 30° to the horizontal with its lower end A on rough horizontal ground, the coefficient of friction being 1/3. The rod rests in contact with a smooth peg C (AC < AB). Find the height of the peg above the ground if the rod is just on the point of slipping.

MOMENTS

So far, we have considered equilibrium under the action of concurrent forces (i.e. forces with **lines of action** all passing through the same point, see Unit 2: Statics I). We shall now examine the conditions for equilibrium under forces that do not necessarily have **lines of action** intersecting at the same point.

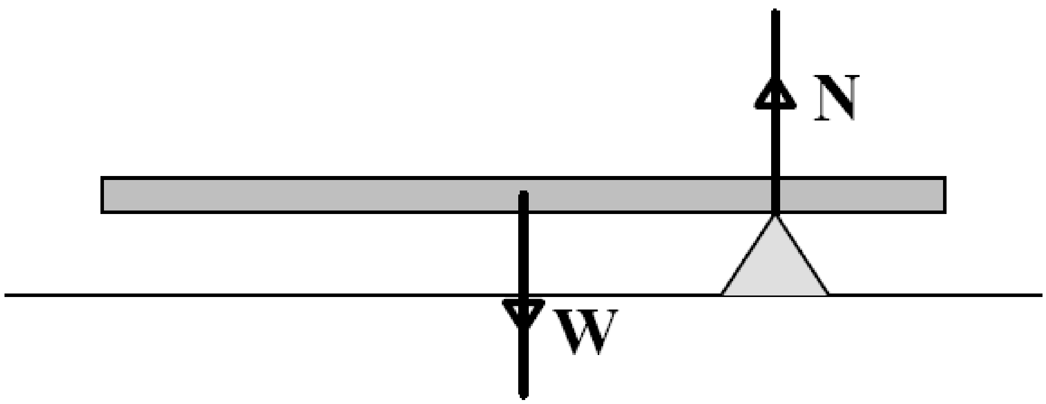
Effect of Non–concurrent Forces

Consider a uniform bar supported at its midpoint:

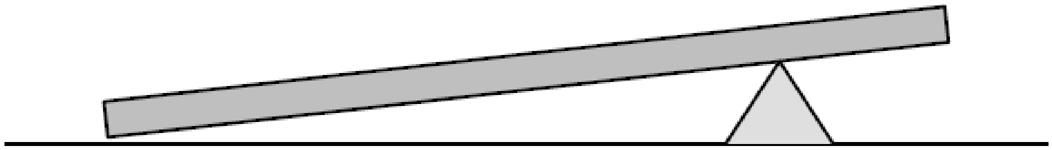


N and **W** are concurrent. **N** + **W** = **0** and the bar is **in equilibrium** .

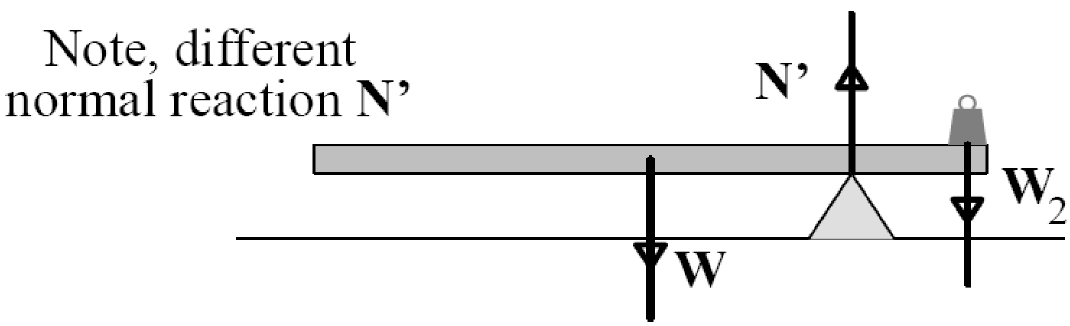
Now move the support to another position:



N and **W** are no longer concurrent. **N** + **W** = **0** as before, but
clearly the bar is not in equilibrium
(and will not remain in the position shown).
The effect of having non–concurrent forces in this instance is that
the bar **rotates** about the support:

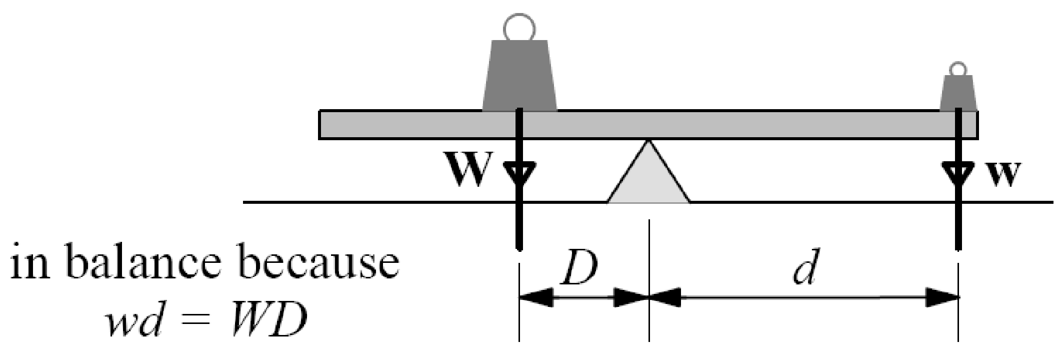


Clearly, if we wish to maintain the bar in equilibrium in its horizontal position, we shall have to counteract the rotation, perhaps by applying an extra weight on the shorter segment:

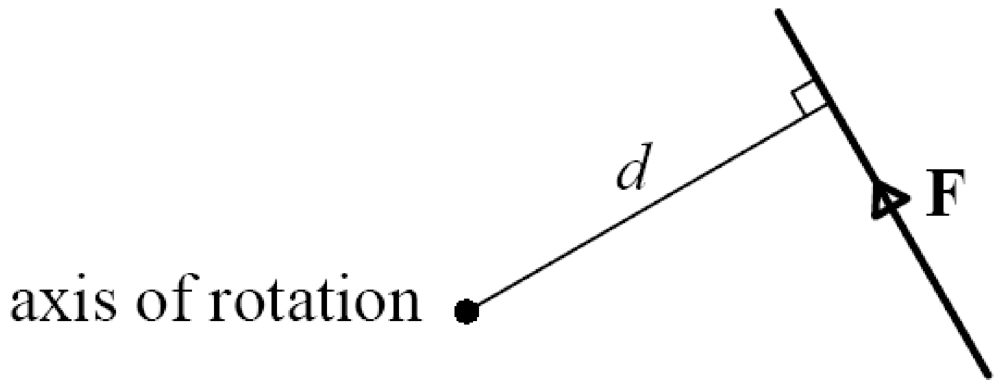


The new condition for static equilibrium is obviously going to involve **position of the forces** as well as the magnitude of the forces, so before we can formulate such a condition, it is

necessary to quantify the effects of the positions of the forces.
An elementary but important result of experience with balancing objects (on scales, see–saws etc) is that both the magnitude of the forces on the object, and position of the forces affect the balance. In fact, it is the product of these two quantities that determines the turning effect:



We therefore define the moment of a force about an axis to be the magnitude of the force multiplied by the perpendicular distance from the axis to the to the line of action of the force :



$$M = F d$$

M is also called:
the **moment** of the force
the **moment of the force** about the axis
the **couple of the force** about the axis

Units of *M*: Nm (note: the same as the joule!)

STATICS EXAMPLE 11

ABCD is a square of side 20 cm. Forces of 50 N, 25 N, 75 N and 100 N act along edges AB, BC, CD and DA respectively. Find the moment of each of these forces about an axis through A and perpendicular to ABCD.

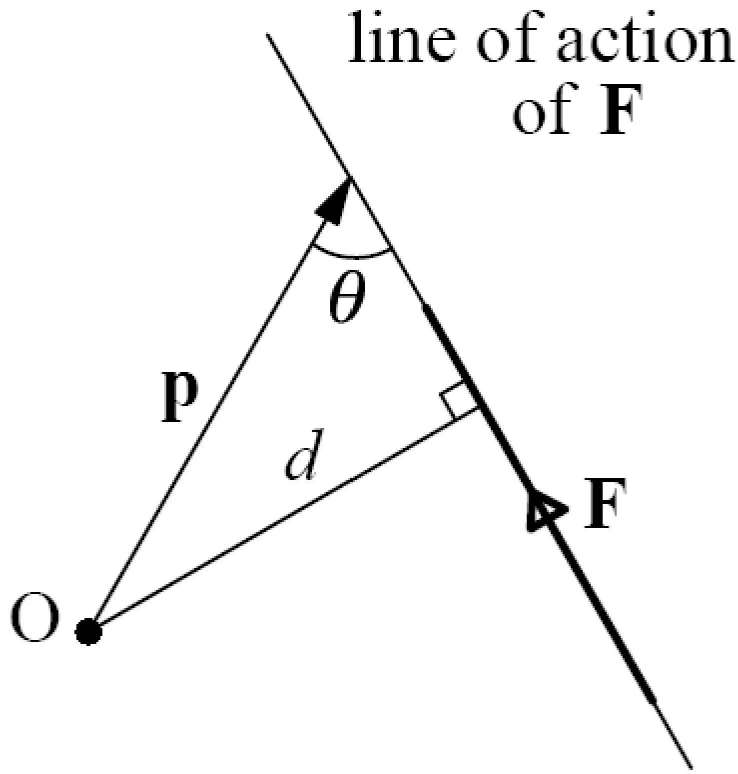
Notes

1. The word 'axis' in the definition of moment of a force does not necessarily refer to a particular axis, like a spindle or the axle of a wheel. It means any line, real or imaginary about which the object might rotate. For example, a spinning football, or the earth, are both rotating about an axis, but this is not a physical object.

2. The definition given for the moment of a force is actuallya special case of the **vector product** of the moment of a force **F** about a point O in space:

$$\mathbf{M} = \mathbf{p} \times \mathbf{F}$$

where **p** is the position vector to any point on the line of action of **F**



From the definition of vector product given in Unit 1: Fundamentals, the moment vector **M** is at **90 degrees** to both **p** and **F** and its magnitude is given by:

$$M = F\, p \sin\theta$$

or: $M = F d$ as before. Usually, the components of **M** would be calculated using the vectors described in Unit 1, which is very straightforward, whatever **p** and **F** might be.

3. In the rest of this unit we shall confine our attention to forces which are co-planar. Any axis of rotation is assumed to be **perpendicular** to this plane. In this case it is possible to refer to moments as the product of force times distance from the axis of rotation (see previous example). This actually means of course **taking** moments about an axis

.

4. A moment can obviously cause rotation in either the positive or the **negative** sense. Positive and negative signs can used to distinguish between these directions. This is useful when **considering** the effect of several moments. In statics, there is no standard convention about whether

moments cause clockwise or anti–clockwise rotation, so it is important to indicate which sign convention is being used when solving problems.

5. The distance used in the calculation of the moment is the perpendicular to the line of action of the force.

STATICS EXAMPLE 12

A ladder of length *L* and weight *W* is inclined at an angle of 60° to the horizontal with its upper end in contact with a smooth vertical wall and its lower end in contact with a rough horizontal surface. What is the moment of the reaction at the upper end about the lower end if the ladder is in equilibrium?

Resultant Moment

The resultant moment about an axis is simply the sum of all the moments about that axis, taking the sign of the moments into account.

STATICS EXAMPLE 13

O is the centre of a square ABCD of side 1 metre. Calculate the magnitudes of three forces which act along AB, BC and CD such that the resultant moment in the sense ABC about A is –1 Nm, about B is +3 Nm and about O is +5 Nm.

STATICS EXAMPLE 14:

The diagram shows a light rod AB in equilibrium. Find the values of F and d.

STATICS EXAMPLE 14a:

A uniform plank AB of length 5m and mass 20kg rests horizontally on two supports, one at each end. A mass of 10kg is positioned on the plank 1m from B. Find the reaction at each of the supports. Take $g=9.8ms^{-2}$.

Principle of Moments

Condition for Equilibrium under Co–planar Forces

For a body to be in equilibrium under a set of **co-planar forces** the **sum of forces in the x and y directions** must be zero, and the sum of moments about any axis perpendicular to the plane of the forces must also be zero.

STATICS EXAMPLE 15

A uniform rod AB of length $2a$ and weight W has its lower end A on rough horizontal ground. It is supported at 30□ to the horizontal by a string attached to a point $a/2$ from B and at right angles to the rod. Find the frictional and normal forces at the ground.

Notes

1. Problems involving co–planar forces have 3 **degrees of freedom** (the body can be displaced in the *x* or *y* directions, or can be rotated about an axis perpendicular to the plane). Any movement can be shown to be expressible as a degrees of freedom. The conditions for static equilibrium can therefore supply no more than 3 independent equations corresponding to these 3 degrees of freedom.

.

2. Typically, the 3 equations of equilibrium will be two relating to the two perpendicular components of force, and one relating to the **moments**, as in the previous example.

3. However, there are alternative choices for the 3 equations,
such as one involving forces, and two involving moments about two different axes.

Equivalent Force Systems

Two **systems of forces** are said to be equivalent if their effect on a body is the same.

For co–planar force systems, this means that their **components** are the same, and their moments are the same (about a given axis).

Just as we defined the resultant of a number of forces acting at a point to be the single force which is equivalent (in the sense given above) to all the others, we can define the resultant of a general set of forces (acting at arbitrary points) to be the equivalent force system.

Again, for co–planar force systems, the **equivalent force system may be a moment** as the following example shows.

STATICS EXAMPLE 16

Forces of $2P$, P, $2P$, $-3P$, $2P$ and $-P$ act along the sides AB, BC, CD, DE, EF and FA respectively of a regular hexagon of side $2a$. Find the magnitude of the resultant force and where the line of action of the force cuts FA, produced if necessary.

Notes

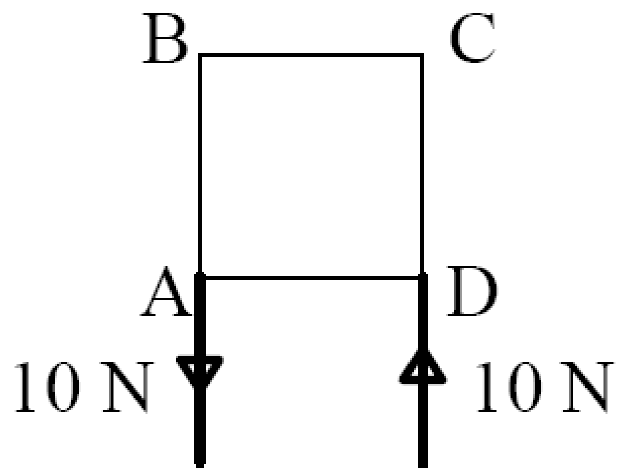
1. We have implicitly used the concept of the resultant of a set of non–concurrent forces when we assumed that the effect of gravity on a body is equivalent to and equal to its total mass acting through its centre of mass.
2. There are 3 pieces of information we need about the resultant of a set of forces: the magnitudes of the two components of the force, and the **direction** of its line of action. Hence we obtain 3 equations as in the previous example.
3. The resultant of a set of co–planar forces is a single force.

STATICS EXAMPLE 17

Forces of 10 N, –20 N, 10 N and –20 N act along the sides AB, BC, CD and DA respectively of a square of side 1 m. Find the resultant of these forces.

Clearly, the square is in equilibrium as there is a resultant (non–zero) moment, yet the resultant of the forces is not zero.
The resultant force system in this case is in fact a **couple**. This is a pure **moment** and can be produced by a pair of **opposite** forces acting along lines a **constant distance** apart.

For example, in the previous case, the resultant could be a pair of forces, one of 10 N in the – *y* direction acting at A, and the other of 10 N in the + *y* direction acting at D:

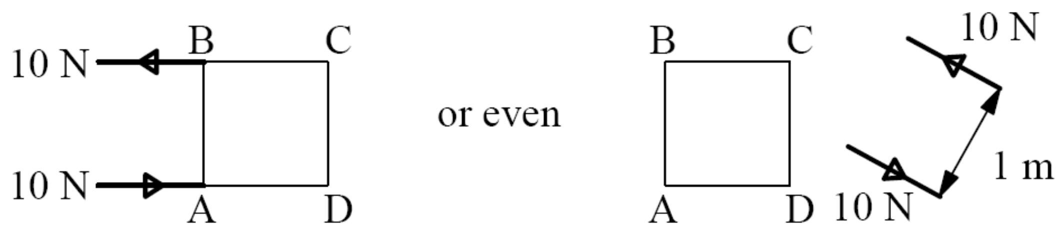


The moment of the couple (sometimes called the **twisting moment** of the couple) is the magnitude of one of the forces **multiplied by** the perpendicular distance between the lines of action of the forces. Like all moments, couples may be positive or negative depending upon direction.

Notes

1. The moment of a couple is the same about any axis perpendicular to the plane in which the couple acts. As a

result, it is immaterial which of the two forces of the couple act, as long as their moment is the same. Thus in the previous example, the couple could equally well have been:



2. Since the couple in the previous example could also have been generated by forces of 5 N acting at a distance of 2 m apart etc, it is clear that the choice of forces to represent a couple is **arbitrary**, and it is usually sufficient just to state the magnitude without referring to forces and moment arms.