ELASTICITY

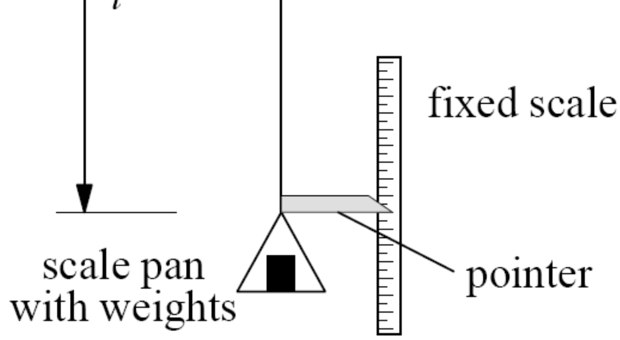
We have studied extensively the relationship between force and the static equilibrium of bodies. In doing this, we have assumed so far that the bodies upon which the forces act are RIGID.

However no real material is perfectly rigid , and all real objects when acted on by a set of forces (in static equilibrium for example) will deform (change their **length**) to some extent, even though this may be very small.

Elasticity is defined as the REVERSIBLE deformation of bodies under load (applied forces). That is to say, a deformation caused by the forces is elastic if the body disappears when the forces are removed.

Elastic deformation is of the utmost importance in the design of artefacts and structures, and the study of statics at more advanced levels is almost entirely devoted to consideration of elastic behaviour (structural mechanics or mechanics of materials).

Elastic behaviour is best illustrated by the stretching of a string or wire. The figure shows how this could be done in practice:



(To be of any use, the pointer/scale would normally incorporate some accurate measurement system or vernier device.)

The natural length of the wire is defined to be its length before the load is applied, usually denoted by l_o. Suppose that a tension T is applied to the wire (by placing an appropriate mass in the scale pan), causing it to stretch to a new length l. The quantity

$$e = l - l_o$$

is called the extension of the wire.

Providing T is not too large (see later) there are two important empirical results:

1. when *T* is removed, the length of the wire returns to l_o (i.e. the deformation is reversible)
2. for most materials, the extension *e* tension *T* and the natural length l_o are proportional .

Symbolically:

$$e \propto l_0 \, T$$

This is more often written as:

$$T \propto \frac{e}{l_0}$$

or:

$$T = \lambda \frac{e}{l_0}$$

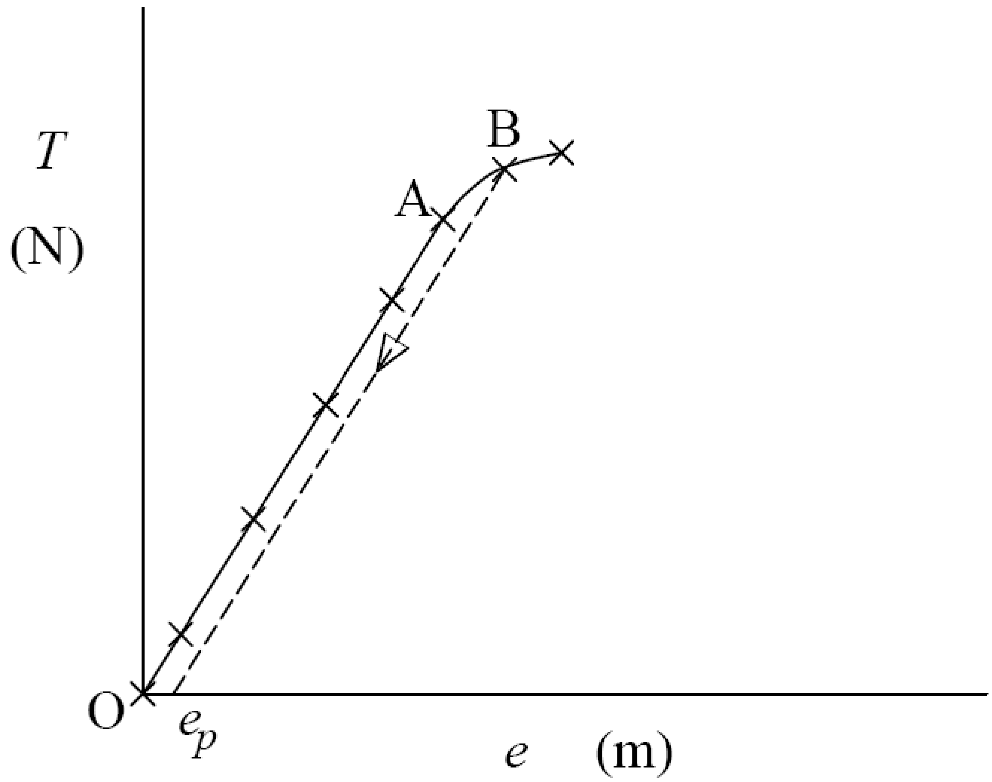
where λ is the modulus of elasticity of the wire.

This last expression is the mathematical statement of HOOKES LAW
Robert Hooke (1635–1703) Professor of Geometry, Gresham College.

Since *e* and l_o have the same units (length) λ has the same units as *T* i.e. Newtons.

Now imagine an experiment in which larger and larger loads are applied to the wire, but with each load being allowed to equilibriate before the next one is applied. The length of the wire is measured after the application of each new load, and after each successive loads.

If the applied loads are plotted on a graph against the extensions produced, something like the following will be obtained:



The experimental measurements (crosses) initially lie on a linear line (OA). The slope of this line is of course

$$\frac{\lambda}{l_0}.$$

Furthermore, when unloaded, the extension disappears.

Eventually, two things will happen:

1. the graph becomes non-linear , i.e. a load is reached beyond which the extension is not proportional to the load (point A)
2. when the wire is unloaded (say from point B, it unloads down the dashed line) the wire reduces in length, but to some value $l_o + e_p$. There has therefore been some permanent extension of the wire. This is called PLASTIC deformation (e subscript p).

Thus there are two different limits:

1. the **limit of proportionality** is the stage in the deformation beyond which deformation is no longer proportional to the applied load i.e. Hooke's law no longer applies (e.g. pointA)
2. the elastic limit (Yield Point) is the stage in the deformation beyond which the original size and shape are not recovered on unloading and permanent deformation occurs (e.g. point B).

Note these limits do not necessarily coincide. If they are distinct, the elastic limit would normally be the limit of proportionality. Many materials behave elastically, even though they may not obey Hooke's law (e.g. Polymers).

Problem Solving

Elastic problems nearly always involve the use of Hooke's law to provide a relationship between force and the amount of deformation in statics or dynamics.

ELASTICITY EXAMPLE 1

A light elastic string of natural length of 4*a* and modulus of elasticity 4 *pg* (where *g* is the acceleration due to gravity) is stretched between two points A and B which are on the same level, where AB = 4*a*. A particle attached to the midpoint of the string hangs in equilibrium with both portions of string making 30 degrees with AB. Find the mass of the particle.

An important characteristic of strings and wires etc is that they can only transmit tensile forces , and that they become slack if the tensile force is reversed.

ELASTICITY EXAMPLE 2

Two light elastic strings AB and BC each of natural length 1 metre and modulus of elasticity $2g$ are joined together at B and have their ends fixed a distance of 3 metres apart in a vertical line, with A below C. Find the mass of the particle that must be attached to B in order that AB becomes slack.

Springs

The ideas formulated for wires etc can be generalised to deal with springs. The difference is that the latter obey Hooke's law in **tension and compression**. We can use the same expression for Hooke's law if we interpret negative values of *e* to mean compression.

Note Hooke's law can be expressed in various ways. For example, sometimes it is convenient to replace the quantity

$$\frac{\lambda}{l_0}$$

by a single parameter *k* (**the stiffness units N/m**).

Other forms use the material property known as YOUNGS MODULUS to relate strain and stress, and extend the law to problems involving PROPERTIES OF MATERIALS. All these alternative forms are beyond the scope of this module.

Elastic Potential Energy

During the extension of a string or wire, or during the extension/compression of a spring, the applied force moves its **POINT OF APPLICATION**. From the principles outlined in a previous unit, we can see that **work** must therefore be done by the applied force in changing the length of the wire or spring. This work done is equivalent to the (**Elastic Potential Energy**) stored in the wire or spring during this operation. Unlike many systems, there is a natural reference point from which to measure the change in **elastic potential energy**, and that is when the wire or spring has its **natural or FREE length**.

Calculation of Elastic Potential Energy

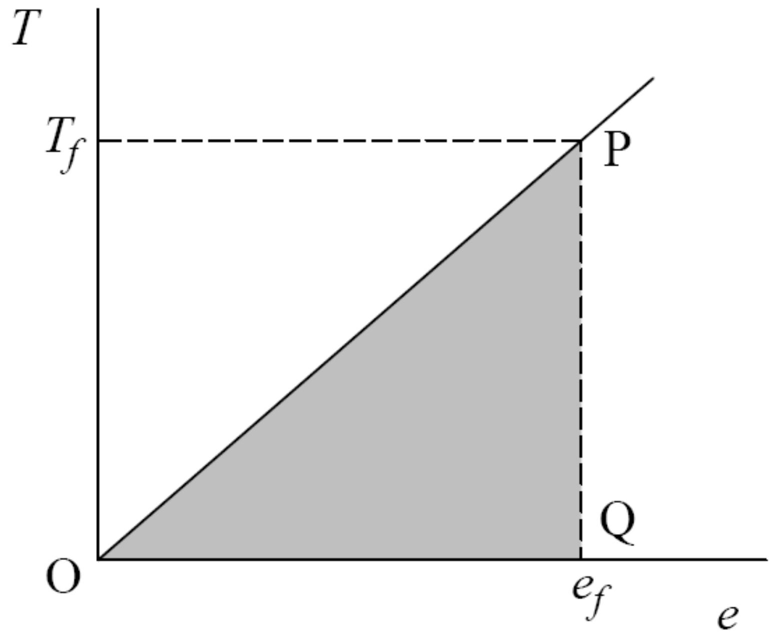
An expression for the potential energy stored in a wire or spring can be derived from an examination of the **deformation of the wire or spring**.

Consider the force required to extend a wire with natural length l_o by an amount *ef* . **Hookes Law** states that:

$$T_f = \lambda \frac{e_f}{l_0}$$

but linear elasticity applies to all stages in the deformation up to this point, so that the applied force must **vary linearly with the extension.**

Graphically:



The work done in **extending the wire or spring** is the shaded area under the straight line, since the work done by a force is the vector product of force with respect to the distance it moves (see Unit: Dynamics I). Thus:

$$\begin{aligned} \text{potential energy} &= \text{Area } \Delta\text{OPQ} \\ &= \frac{1}{2}\text{PQ} \;.\; \text{OQ} \\ &= \frac{1}{2} T_f \;.\; e_f \end{aligned}$$

but:

$$T_f = \lambda \frac{e_f}{l_0}$$

so:

$$\begin{aligned} \text{potential energy} &= \frac{1}{2}\lambda \frac{e_f}{l_0} e_f \\ &= \frac{1}{2}\lambda \frac{e_f^2}{l_0} \end{aligned}$$

To summarise, the elastic potential energy stored in a wire of natural length l_o and modulus of elasticity λ when extended by amount *e* is:

$$\text{EPE} = \frac{1}{2}\lambda \frac{e^2}{l_0}$$

Note as a consequence of the reversibility of length on unloading, the potential energy stored in an elastic wire is completely recovered when the load is removed, and may be converted to another form of ENERGY (see Unit : Dynamics I).

The previous expression and remarks apply equally well to springs, except that in these EPE may be stored in **tension** or compression.

Note since the expression for EPE involves the square of *e* , the value is always **positive** whether *e* is positive (**tension**) or negative (compression).